MIT PRIMES STEP JUNIOR GROUP

PRIMES Conference, May 14th, 2023

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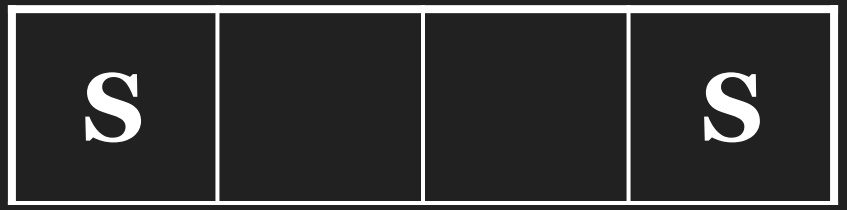
INTRODUCTION

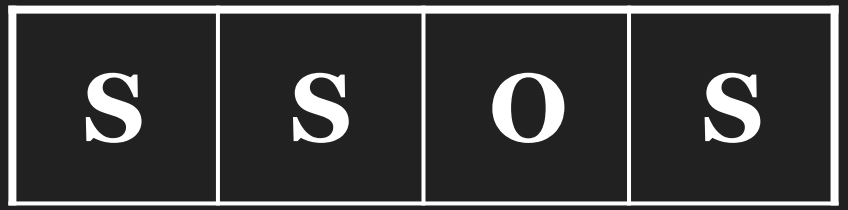
WHAT IS **SOS?**

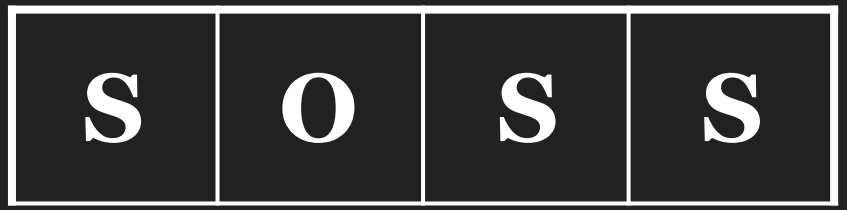
The Y2K Game is played on a 1 × 2000 grid as follows: Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.
Source: 1999 USAMO

- Before we see the solution, let's check out some smaller boards!

WHAT WOULD A GAME OF **SOS** LOOK LIKE?







WHO WANTS TO TRY A GAME OF **SOS?**

LOSING PAIRS

- Losing pairs are a set of two consecutive empty spaces surrounded by letters such that placing any letter in one of those empty spaces is an immediate win for the opponent.
- In our example:
  - The formation of 

is such that any move by the first player leads to a victory for the second player.
- When is it optimal to create a losing pair?
- The player who sees an odd number of empty squares before their turn will never see only the two squares of a losing pair.

SAFE SQUARE

- How do we know that someone can't win outside of a losing pair?
- The person who sees an odd number of empty squares has a guaranteed safe square at all times
- For the game of SOS the proof of guaranteed safe squares is as follows:
- Since there are an odd number of squares, there must be one of the following:
  - One square surrounded by letters, in which we can place an O safely.
  - Three or more empty squares in a row, in which we can place an O in the center.

THE ANSWER

If the first player puts an S down, the second player can create a losing pair. Suppose the first player puts an O somewhere. There still exists a strip of seven empty cells not neighboring O. The second player places an S in the middle of this strip. The next time the second player moves, he can place the second S for the losing pair that doesn't neighbor the given O. Such a move doesn't create an immediate win for the first player. After the losing pair is created, the second player wins because they see an odd number of squares and will always have a safe move outside a losing pair. Thus, the second player will win with optimal play.

Source: 1999 USAMO

THE END!

That's it. The slideshow is *DEFINITELY* over. Thank you for coming to our presentation.

SOS VARIANTS

Tricked you, didn't we?

We can change the target string!

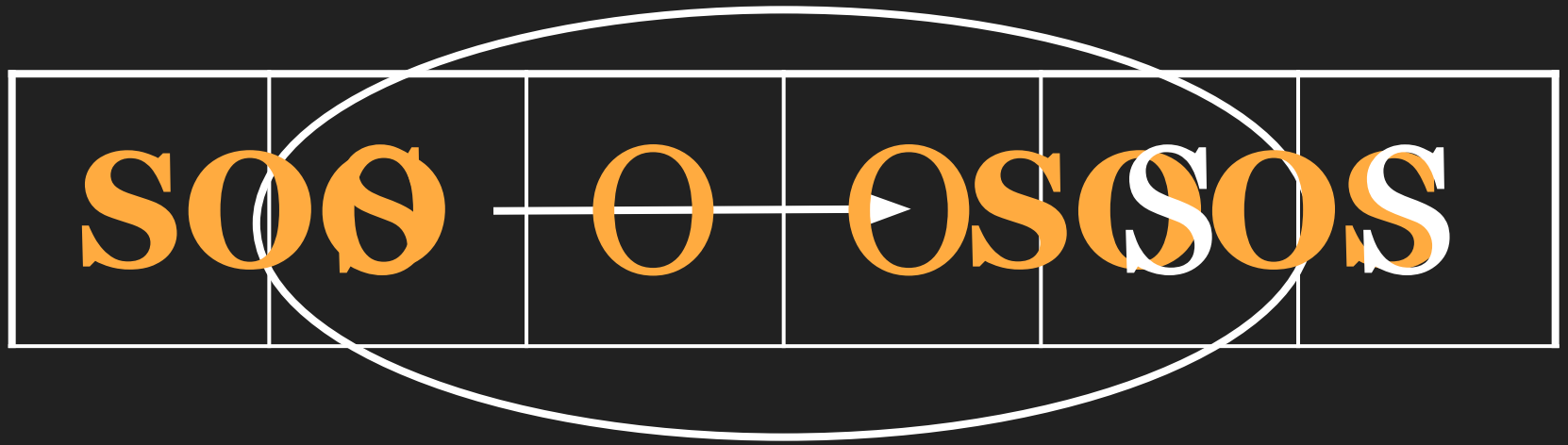
Say… SOO?

SOS VARIANTS: SOO

- It turns out that both players can implement a simple drawing strategy:
  - If, on your turn, you can win, you do so.
  - Otherwise, you place an S in the rightmost square available.

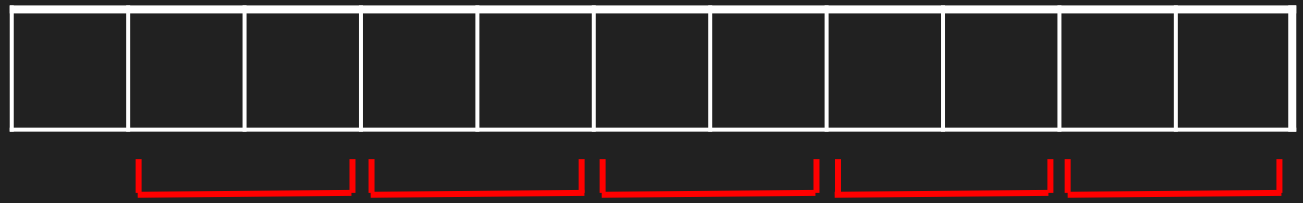
SOS VARIANTS: SOO

SOS VARIANTS: SOO



DOMINOES

We can divide the board into dominos, which is the idea of combining every two consecutive blocks and viewing them as one. If there is an odd number of squares, let the leftmost square stand on its own. This is an important idea in many variants.



STRINGS CONTAINING SSS, PART 1

Strategy:

- On their turn, if a player can win, then the player does so.

- Otherwise, if a domino exists with exactly one of the squares filled with a letter, the player fills the other square with the other letter. If not, the player places an O anywhere. After the player's turn, any S inside a domino has to be next to an O.

STRINGS CONTAINING SSS, PART 2

- If two consecutive S's exist in the leftmost part of the board, the board will start with SSO, as the second S will be part of a domino. This also means the board has to be odd, since two S's cannot be part of the same domino.

- The only other possible case to have two consecutive S's is the OSSO substring, where the string is a combination of two dominos.

STRINGS CONTAINING SSS, PART 3

Having an S_S substring at any point in the game is impossible, since the blank square will be in at least one of the dominoes containing an S, which means a player should've placed an O in the empty space. Therefore, there are no situations when one move can create an SSS substring, so all games containing them are drawn.

SOSO, PART 1

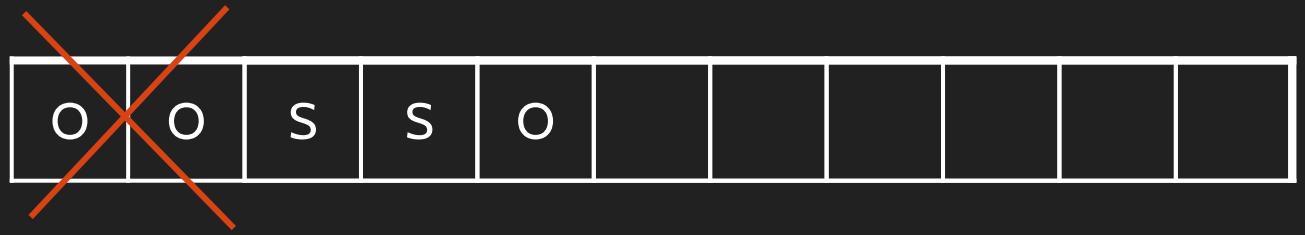
Here is a very important idea that cuts our work in half:

- To prove a target string is always a draw, we simply need to prove the second player can guarantee a draw.
- This is because the first player can place an O in the leftmost available square.

PURGING

To assist us in our proof we introduce the notion of *purging*:

- Purging is defined as removing the leftmost two filled cells.
- This can only be done if these two cells can't contribute to creating SOSO.



SOSO, PART 2

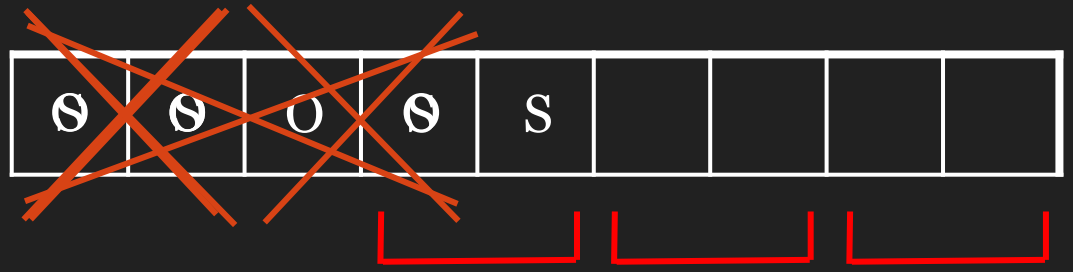
Using the results from Part 1 we only need to consider the second player.

On an even board,

- We use dominoes

On an odd board. Mark the leftmost square, split the others into dominoes. Consider when the board starts with:

- OO
- S_O
- SO
- SS(S/O)(S/O)
- SSO_ _



2-BY-N GRID

We expand the grid to a 2-by-N grid. Though it is 2-dimensional, there is not enough space to make a vertical or diagonal SOS. Because there are always an even amount of squares, player 2 will win if a losing pair is created because player 2 can always use the safe square strategy.

It turns out that player 1 will never win on a 2-by-N grid. Player 2 can always reflect player 1's last move to the other board if it was a safe move. If player 1 had a safe move, player 2 copies it. Otherwise, player 2 wins if a losing pair is on the board, which they are able to force if N is greater or equal to 7.

MISÈRE GAMES

In Misère games, you lose if you make the target string. For SOS, this is pretty easy. Take any box. If placing an O loses, the box is surrounded by S's, but in that case you can place an S in that square.

S		S

Thus, the game is drawn as every square can be played in safely.

LOOP-AROUND

The board loops around now. This doesn't prevent any of the winning strategies, but it does make it easier for player 2 to make a losing pair. It gives player 2 the chance to make a losing pair on a 1-by-10 board, thus implementing the same strategies on a smaller board. It also allows player 1 a winning strategy on 1-by-3 board.

RESULTS FROM PROGRAM

We wrote a program that takes a board size and the target strings and produces the result of the game. Thus, we have a whole table of results of games that we have solved and those we haven't solved.

1=P1 wins
0.5=Draw
0=P2 wins

	SOS	SSS	SOO	SOS-OSO	SSS-OOO	SOO-OOS
1x3:	0.5	0.5	0.5	0.5	0.5	1
1x4:	0.5	0.5	0.5	0.5	0.5	0.5
1x5:	0.5	0.5	0.5	0.5	0.5	1
1x6:	0.5	0.5	0.5	0	0	1
1x7:	1	0.5	0.5	1	1	1
1x8:	0.5	0.5	0.5	0	0	0.5
1x9:	1	0.5	0.5	1	1	1
1x10:	0.5	0.5	0.5	0	0	1
1x11:	1	0.5	0.5	1	1	1
1x12:	0.5	0.5	0.5	0	0	0.5
1x13:	1	0.5	0.5	1	1	1
1x14:	0.5			0	0	0.5
1x15:	1					1
1x16:						1

ACKNOWLEDGEMENTS

Thank you to the MIT PRIMES STEP program and Dr. Tanya Khovanova for providing us with this opportunity.

SPECIAL THANKS TO:

Our Family and Friends,

Especially our Parents (Happy Mother's Day!!)

THANK YOU!!!

Any Questions?

WORKS CONSULTED

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IMAGE CREDITS

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